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On quasifactorability in graphs

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Abstract

Given a graph G and two functions f and $g: V(G) \to \mathbb{Z}^+$ with $f(v) \geqslant g(v)$ for each $v \in V(G)$, a (g, f)-quasifactor in G is a subgraph Q of G such that for each vertex v in $V(Q)$, $g(v) \leqslant d_Q(v) \leqslant f(v)$; in the particular case when $\forall v \in V(Q)$, $f(v) = g(v) = k \in \mathbb{N}$, we say that Q is a k-quasifactor. A subset S of vertices of G is said (g, f)-quasifactorable in G if there exists some (g, f)-quasifactor that contains all the vertices of S. In this paper, we give several results on the 2-quasifactorability of a vertex subset which are related to minimum degree, degree sum, independence number and neighborhood union conditions. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction and notations

The graphs we consider in this paper are finite, undirected, without loops nor multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges. For A and B two disjoint subsets of $V(G)$, $e(A, B)$ denotes the number of edges with one extremity in A and the other in B. Given a subgraph K of G, we often write K instead of $V(K)$ if there is no ambiguity. If x is a vertex of G, $N_K(x)$ is the neighborhood of x in K and $d_K(x) = |N_K(x)|$. If $K = G$, we often write $N(x)$ and $d(x)$ instead of $N_G(x)$ and $d_G(x)$. Also we write $\delta(K)$ for the minimum of $\{d_G(x),\ x \in K\}$, $\sigma_3(K)$ for the minimum degree sum of any three independent vertices in K, $n_3(K)$ for the minimum cardinality of the neighborhood union of any three independent vertices in K and $\alpha(K)$ for the independence number of the subgraph K. If K' is another subgraph of G, $K - K'$ represents the vertices of K that do not belong to K'. Given $t \in \mathbb{R}^+$, the graph G is t-tough if $|S| \geqslant t\omega(G - S)$ for every subset S of $V(G)$ with $\omega(G - S) > 1$, where $\omega(G - S)$ is the number of components of $G - S$. The toughness of G, denoted by $\tau(G)$, is the largest value of t such that G is t-tough.

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For $k \in \mathbb{N}$, a k-factor of G is a spanning k-regular subgraph of G. This notion has been generalized in the following way [3]: given two functions f and $g: V(G) \to \mathbb{Z}^+$ with $f(v) \geqslant g(v)$ for each $v \in V(G)$, a (g, f)-factor in G is a spanning subgraph Q of G such that $g(v) \leqslant d_Q(v) \leqslant f(v)$ for each vertex v in G. If we suppress the constraint that Q is spanning, we say that the subgraph Q is a (g, f)-quasifactor. More precisely, a (g, f)-quasifactor in G is a subgraph Q of G such that $g(v) \leqslant d_Q(v) \leqslant f(v)$ for each vertex v in $V(Q)$. In the particular case when $\forall v \in V(Q)$, $f(v) = g(v) = k \in \mathbb{Z}^+$, we just say that Q is a k-quasifactor. A subset S of vertices of G is said to be (g, f)-quasifactorable in G if there exists some (g, f)-quasifactor containing all the vertices of S. Other notations can be found in [1,2,6].

In the last years there have been many results on hamiltonian cycles, i.e. cycles through every vertex of a given graph. Recently, people have been interested into the more general property of cyclability of a set of vertices. A subset S of vertices of the graph G is said to be cyclable in G if there is some cycle of G that contains all the vertices of S. Many sufficient conditions insuring cyclability have been proved (see for example [4]). Those conditions applied to $S = V(G)$ give as corollaries well-known sufficient conditions for hamiltonicity.

In this paper, instead of hamiltonian cycles, we are interested in (g, f)-factors and more particularly in 2-factors that constitute a generalization of hamiltonian cycles. We then look for the analogous property of cyclability, that is, given a subset S of vertices of G, we look for a 2-quasifactor that contains all the vertices of S. If G has a 2-quasifactor containing all vertices of S, we say that S is 2-quasifactorable in G.

Remark 1. If G is 2-connected, which will necessarily be the case in the remaining of this paper since G is 1-tough, we do not have to care for $|S| = 1$ or 2 since any two vertices are in a cycle.

In next section, we will prove the following theorems.

Theorem 1. *Given $t \in \mathbb{R}$, $1 \leqslant t < 2$, let G be a t-tough graph of order n and S a subset of $V(G)$ of order at least 3. Then S is 2-quasifactorable in G provided one of the following three conditions holds*:

(1) $\delta(S) \geqslant n(2 - t)/(1 + t)$.
(2) $\sigma_3(S) \geqslant 3n(2 - t)/(1 + t) + 2$.
(3) $n_3(S) \geqslant n(2 - t)/(1 + t) + 6$.

Theorem 2. *Let G be a 1-tough graph and S a subset of $V(G)$ satisfying $\delta(S) > \alpha(S)$. Then S is 2-quasifactorable in G.*

The immediate corollary is the following.

Theorem 3. *Let G be a 1-tough graph and S a subset of $V(G)$ satisfying $\delta(S) > |S|$. Then S is 2-quasifactorable in G.*

Notice that the condition $\delta(S) > \alpha(S)$ is best possible as shown by the graph G_0 defined in the following way: $V(G_0) = X \cup S \cup C_1 \cup C_2 \cup \cdots \cup C_{2x+2}$; $|X| = x$, $S = \{s_1, s_2, \ldots, s_{x+2}\}$ is an independent set with cardinality exactly $x+2$, every vertex $x \in X$ is adjacent to all the other vertices, $C_1, C_2, \ldots, C_{2x+2}$ are disjoint cliques of $G_0 - (X \cup S)$, s_1, s_2 and s_3 are joined to distinct vertices of C_1, s_i is joined to C_{i+1} for $1 \leqslant i \leqslant 3$ and s_i is joined to C_{2i-3} and C_{2i-2} for $4 \leqslant i \leqslant x+2$. Clearly, G_0 is 1-tough, $\delta(S) = \alpha(S)$ but S is not 2-quasifactorable in G_0.

2. Preliminary results

Let us first give some additional notations, where G is a graph and f_0 any function of $V(G)$ in $\mathbb{Z}^+$. For a subset U of $V(G)$, we write $f_0(U) = \sum_{u \in U} f_0(u)$ with $f_0(\emptyset) = 0$. In particular $d_G(U) = \sum_{u \in U} d_G(u)$.

We denote the set of all the ordered pairs (X, Y) of subsets of $V(G)$ by $\mathscr{B}(G)$ such that $X \cap Y = \emptyset$. Given $B = (X, Y) \in \mathscr{B}(G)$, P a subset of $V(G)$ and f and g two functions on $V(G)$ in $\mathbb{Z}^+$, let $h(B; f, g, P)$ denote the number of components C of $G - (X \cup Y)$ for which $C \subseteq P$ and $f(C) + e(C, Y) \equiv 1 \pmod 2$. We then define

$$\delta(B; f, g, P) = h(B; f, g, P) - f(X) - d_G(Y) + g(Y) + e(X, Y).$$

A component C in $G - (X \cup Y)$ such that $e(C, Y) \equiv 1 \pmod 2$ is called Y-odd component and the components of $G - (X \cup Y)$ that are not Y-odd are said Y-even. Also, $\omega_Y^1(G - (X \cup Y))$ is the number of Y-odd components in $G - (X \cup Y)$.

Given a graph G and two functions f and $g : V(G) \to \mathbb{Z}^+$ with $f(v) \geqslant g(v)$ for each $v \in V(G)$, let us recall the Lovász's parity theorem ([3,5]) concerning (g, f)-factors in G.

Theorem 4. *Let G be a graph, f and g : $V(G) \to \mathbb{Z}^+$ two functions such that $f(v) \geqslant g(v)$ for each $v \in V(G)$, and P a subset of $V(G)$ such that $\{v \in V(G) \mid f(v) = g(v)\} \subseteq P$ and $f(v) \equiv g(v) \pmod 2$ for each v in P. Then G contains a (g, f)-factor F such that $d_F(v) \equiv f(v) \pmod 2$ for each v in P if and only if $\delta(B, f, g, P) \leqslant 0$ for all $B \in \mathscr{B}(G)$.*

As a corollary of Theorem 4, we get the following proposition.

Proposition 1. *Let G be a graph and S a subset of $V(G)$. Then S is 2-quasifactorable in G if and only if for any pair $(X, Y) \in \mathscr{B}(G)$*

$$\omega_Y^1(G - (X \cup Y)) - 2|X| + 2|Y \cap S| - \sum_{y \in Y} d_{G-X}(y) \leqslant 0.$$

Proof. Proposition 1 is a direct consequence of Theorem 4 by choosing P, f, g as follows: $P = V(G)$, $f(v) = 2$ for any $v \in V(G)$, $g(v) = 2$ if $v \in S$ and $g(v) = 0$ otherwise. $\square$

Using Proposition 1 and parity arguments, we get the following.

Corollary 1. *Let G be a graph and S a subset of $V(G)$. The following two assertions are equivalent*:

(1) *S is not 2-quasifactorable in G.*
(2) *There exists a pair (X,Y) in $\mathscr{B}(G)$ such that*

$$\xi_G(X,Y;S) = \omega_Y^1(G-(X\cup Y)) - 2|X| + 2|Y\cap S| - \sum_{y\in Y} d_{G-X}(y) \geqslant 2.$$

In the end of this section, we consider a graph G with order $n\geqslant 3$, toughness $t\geqslant 1$ and that contains a subset S of $V(G)$ not 2-quasifactorable in G. Moreover, we suppose that the following two properties (A) and (B) hold:

(A) The number of edges in G is maximum.
(B) The pair (X,Y) in $\mathscr{B}(G)$, associated to S by Corollary 1 such that

$$\xi_G(X,Y;S) = \omega_Y^1(G-(X\cup Y)) - 2|X| + 2|Y\cap S| - \sum_{y\in Y} d_{G-X}(y) \geqslant 2$$

also satisfies

B(1) $\xi_G(X,Y;S)$ is maximum.
B(2) Y has cardinality minimum subject to B(1).
B(3) X has cardinality maximum subject to B(2).

Remark 2. From the definition of $\xi_G(X,Y;S)$ and assumption (A), clearly the subgraph induced by X and the components of $G-(X\cup Y)$ are complete subgraphs of G.

We now give five claims to specify the structure of G, especially the edge repartition. *Claim* 1. *Y is independent and contained in S. Moreover, for every $y\in Y$ and every component C* in $(G-(X\cup Y))$, *$N_C(y)=0$ except possibly if C is Y-odd, in which case $N_C(y)$ is at most* 1.

Proof. By B(1), B(2) and parity arguments, we necessarily have for any $y_0\in Y$

$$\xi_G(X,Y;S) \geqslant \xi_G(X,Y-\{y_0\};S) + 2,$$

that is

$$\begin{aligned}&\omega_Y^1(G-(X\cup Y)) - 2|X| + 2|Y\cap S| - \sum_{y\in Y} d_{G-X}(y)\\ &\quad\geqslant 2 + \omega^1_{Y-\{y_0\}}(G-(X\cup(Y-\{y_0\}))) - 2|X| + 2|(Y-\{y_0\})\cap S|\\ &\quad - \sum_{y\in Y-\{y_0\}} d_{G-X}(y).\end{aligned} \tag{1}$$

Putting $\chi_S(y_0)=1$ if $y_0\in S$ and $\chi_S(y_0)=0$ otherwise, inequality (1) implies

$$\omega_Y^1(G-(X\cup Y)) - \omega^1_{Y-\{y_0\}}(G-(X\cup(Y-\{y_0\}))) \geqslant 2 - 2\chi_S(y_0) + d_{G-X}(y_0).$$

The difference $\omega_Y^1(G-(X\cup Y))-\omega_{Y-\{y_0\}}^1(G-(X\cup(Y-\{y_0\}))$ is also less than or equal to $u(y_0)$ where $u(y_0)$ is the number of Y-odd components C in $G-(X\cup Y)$ such that $N(y_0)\cap C\neq\emptyset$. We then get

$$\begin{aligned}2-2\chi_S(y_0)+d_{G-X}(y_0) &\leqslant \omega_Y^1(G-(X\cup Y))-\omega_{Y-\{y_0\}}^1(G-(X\cup(Y-\{y_0\})))\\ &\leqslant u(y_0)\leqslant d_{G-X}(y_0)\end{aligned}$$

which gives

$$0\leqslant 2-2\chi_S(y_0)\leqslant 0$$

and necessarily

$$\chi_S(y_0)=1 \quad \text{and so } d_{G-X}(y_0)=u(y_0).$$

Clearly, those equalities imply that y_0 belongs to S, has no neighbor in Y and has at most one neighbor in a given component of $G-(X\cup Y)$, moreover only in the case when this component is Y-odd. Since this is true for any y_0 in Y, Claim 1 is proved. □

From Corollary 1 and Claim 1, we have

$$2\leqslant \xi_G(X,Y;S)=\omega_Y^1(G-(X\cup Y))-2|X|+2|Y|-\sum_{y\in Y} d_{G-X}(y). \tag{2}$$

Claim 2. $|Y|\geqslant 3$.

Proof. First from (2) and $\omega_Y^1(G-(X\cup Y))\leqslant\sum_{y\in Y} d_{G-X}(y)$ we have

$$2\leqslant\omega_Y^1(G-(X\cup Y))-2|X|+2|Y|-\sum_{y\in Y} d_{G-X}(y)\leqslant -2|X|+2|Y|,$$

whence

$$|Y|\geqslant|X|+1.$$

Assume now $|Y|\leqslant 2$. From 2-connectivity we know that $d(y)\geqslant 2$ for every vertex y in Y.

Case 1. $|X|=0$ and $|Y|=1$ or $|X|=1$ and $|Y|=2$. In both cases, we get the following from (2):

$$2\leqslant\omega_Y^1(G-(X\cup Y))+2-\sum_{y\in Y} d_{G-X}(y)$$

and so

$$\sum_{y\in Y} d_{G-X}(y)\leqslant\omega_Y^1(G-(X\cup Y))\leqslant\sum_{y\in Y} d_{G-X}(y),$$

i.e.

$$\omega_Y^1(G-(X\cup Y))=\sum_{y\in Y} d_{G-X}(y).$$

It results that $\omega_Y^1(G-(X\cup Y))$ is at least 2 and no odd component in $(G-(X\cup Y))$ can be adjacent to more than one vertex in Y. Corresponding graphs clearly contradict that toughness is at least 1.

Case 2. $|X|=0$ and $|Y|=2$. Then in this case inequality (2) implies $2\leqslant\omega_Y^1(G-(X\cup Y))+4-\sum_{y\in Y}d_{G-X}(y)$ and, from $\omega_Y^1(G-(X\cup Y))\leqslant|X\cup Y|=2$, $\sum_{y\in Y}d_{G-X}(y)-2\leqslant\omega_Y^1(G-(X\cup Y))\leqslant 2$.

The only possible case is $\omega_Y^1(G-(X\cup Y))=2$, and both vertices in Y have exactly two neighbors in $G-(X\cup Y)$ (one neighbor in each Y-odd component). Then each Y-odd component is Y-even, a contradiction. □

Claim 3. Every vertex $u\in G-(X\cup Y)$ satisfies $e(u,Y)\leqslant 1$.

Proof. Assume that $u\in G-(X\cup Y)$ satisfies $e(u,Y)\geqslant 2$. Let $X^*=X\cup\{u\}$. From Claim 1, u belongs to an Y-odd component and so ω_Y^1 remains unchanged or decreases by 1. Assumption B(3) in the choice of (X,Y) implies $\xi_G(X\cup\{u\},Y;S)\leqslant\xi_G(X,Y;S)-2$, that is

$$\omega_Y^1(G-(X\cup\{u\}\cup Y))-2|X\cup\{u\}|+2|Y|-\sum_{y\in Y}d_{G-(X\cup\{u\})}(y)$$
$$\leqslant\omega_Y^1(G-(X\cup Y))-2|X|-2+2|Y|-\sum_{y\in Y}d_{G-X}(y).$$

Thus,

$$2\leqslant e(u,Y)\leqslant\omega_Y^1(G-X\cup Y)-\omega_Y^1(G-((X\cup\{u\})\cup Y))\leqslant 1,$$

a contradiction. □

Claim 4. $\max\{\delta(S),(\sigma_3(S)-2)/3,n_3(S)-6\}\leqslant|X|+2$.

Proof. Since $|Y|\geqslant 3$, it makes sense to minimize $\sum_{y\in Y}d_{G-X}(y)$ with the help of $\sigma_3(S)$ or $n_3(S)$.

We first have that $\sum_{y\in Y}d_{G-X}(y)<3|Y|$ since otherwise inequality (2) implies

$$2\leqslant\omega_Y^1(G-(X\cup Y))-2|X|+2|Y|-3|Y|,$$

that is

$$|X|+|Y|\leqslant\omega_Y^1(G-(X\cup Y))-2,$$

contrary to 1-toughness.

If $\delta(S)>|X|+2$, then $d_{G-X}(y)\geqslant 3$ for every vertex $y\in Y$ and so $\sum_{y\in Y}d_{G-X}(y)\geqslant 3|Y|$, a contradiction.

By considering any triple in Y and by the definition of $\sigma_3(S)$ and $n_3(S)$, we have

$$\sum_{y\in Y}d_{G-X}(y)\geqslant\frac{\sigma_3(S)-3|X|}{3}|Y|\quad\text{and}\quad\sum_{y\in Y}d_{G-X}(y)\geqslant\frac{n_3(S)-|X|}{3}|Y|.$$

If either $(\sigma_3(S)-2)/3 > |X|+2$ or $n_3(S)-6 > |X|+2$, we can deduce $\sum_{y\in Y} d_{G-X}(y) \geqslant 3|Y|$, also a contradiction. □

Before proving the last claim, we need the following construction and observations. Call $C_1, C_2, \ldots, C_s$ the Y-odd components of $G-(X\cup Y)$ with $s=\omega_Y^1(G-(X\cup Y))$ and choose $a_i \in C_i \cap N(Y)$, $1 \leqslant i \leqslant s$. Put $C_i' = C_i \cap N(Y) - \{a_i\}$, $1\leqslant i\leqslant s$ and $F = X \cup (\bigcup_{i=1}^{s} C_i')$. We have

$$t \leqslant \frac{|F|}{\omega(G-F)} \leqslant \frac{|X| + e(Y, G-(X\cup Y)) - s}{|Y|}$$
$$\leqslant \frac{|X| + \sum_{y\in Y} d_{G-X}(y) - \omega_Y^1(G-(X\cup Y))}{|Y|} \leqslant \frac{2|Y|-|X|-2}{|Y|}. \tag{3}$$

Inequality (3) implies

$$|X| + 2 \leqslant (2-t)|Y|. \tag{4}$$

Claim 5. $\sum_{y\in Y} d(y) \leqslant |Y|^2$.

Proof. From (2) and 1-toughness that implies $\omega_Y^1(G-(X\cup Y)) \leqslant |X|+|Y|$, we have

$$e(Y, G-(X\cup Y)) = \sum_{y\in Y} d_{G-X}(y) \leqslant -2 + \omega_Y^1(G-(X\cup Y))$$
$$-2|X| + 2|Y| \leqslant 3|Y| - |X| - 2. \tag{5}$$

Since $t \geqslant 1$, inequality (4) gives

$$|Y| \geqslant |X| + 2. \tag{6}$$

Summing the degrees on Y and using (5) and (6), we now get

$$\sum_{y\in Y} d(y) \leqslant |X||Y| + e(Y, G-(X\cup Y)) \leqslant |X||Y| + 3|Y| - |X| - 2$$
$$= (|X|+3)(|Y|-1) + 1 \leqslant (|Y|+1)(|Y|-1) + 1 = |Y|^2. \qquad \square$$

3. Proofs of Theorems 1 and 2

For both theorems we proceed by contradiction, i.e. we assume that the graph G and the subset S of $V(G)$ satisfy the assumptions of the respective theorem but S is not 2-quasifactorable in G. We also assume that G and the pair (X, Y) associated to S by Corollary 1 fulfill properties (A) and (B) defined in Section 1 whence we can use inequalities and claims of Section 1.

Proof of Theorem 1. Assume $|Y| \geqslant n/(1+t)$, by using (3) and Claim 3, we have

$$t \leqslant \frac{|X| + e(Y, G-(X\cup Y)) - s}{|Y|} \leqslant \frac{|X| + |G-(X\cup Y)| - s}{|Y|}$$

$$= \frac{n - |Y| - \omega_Y^1(G - (X \cup Y))}{|Y|} \leqslant \frac{n - n/(1+t) - \omega_Y^1(G - (X \cup Y))}{n/(1+t)}$$
$$\leqslant t - \frac{(1+t)\omega_Y^1(G - (X \cup Y))}{n},$$

a contradiction. Hence we have $|Y| < n/(1+t)$.

By the conditions (1), (2) and (3) respectively and by using Claim 4 and (4), we deduce, respectively, that

$$\frac{2-t}{1+t} n \leqslant \delta(S) \leqslant |X| + 2 \leq (2-t)|Y|,$$
$$\frac{2-t}{1+t} n \leqslant \frac{\sigma_3(S) - 2}{3} \leqslant |X| + 2 \leqslant (2-t)|Y|,$$
$$\frac{2-t}{1+t} n \leqslant n_3(S) - 6 \leqslant |X| + 2 \leqslant (2-t)|Y|.$$

All those give $|Y| \geqslant n/(1+t)$, a contradiction that achieves the proof of Theorem 1. □

Proof of Theorem 2. Since $Y \subset S$ is independent, it comes by Claim 5

$$\delta(S)|Y| \leqslant \sum_{y \in Y} d(y) \leqslant |Y|^2 \leqslant \alpha(S)|Y|,$$

a contradiction with our assumption. □

Notice that Theorem 3 can be obtained as a corollary of Theorem 2 or directly of Claim 5.

References

[1] D. Bauer, E. Schmeichel, Toughness, minimum degree and the existence of 2-factor, J. Graph Theory 18 (1994) 241–256.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.
[3] M. Cai, On some factor theorems of graphs, Discrete Math. 98 (1991) 223–229.
[4] O. Favaron, E. Flandrin, H. Li, Y. Liu, F. Tian, Z. Wu, Sequences, claws and cyclability of graphs, J. Graph Theory 21 (1996) 357–369.
[5] L. Lovász, The factorization of graphs II, Acta Math. Acad. Sci. Hungar. 23 (1972) 223–246.
[6] W. Tutte, The factors of graphs, Canad. J. Math. 4 (1953) 314–328.